HE 215 : Nuclear & Particle Physics Course

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Outline I

- Quantum Electrodynamics
  - The Dirac Equation
  - Solutions to the Dirac Equation
  - Plane-Wave Solutions to the Dirac Equation
  - Transformation Properties of Spinors
  - The Feynman Rules for QED
  - QED Catalogue
  - Casimir's Trick and Trace Theorems
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Quantum Electrodynamics

This is chapter 7 in Griffiths.

Quantum Electrodynamics

In Chapter 6 we used ABC Toy theory to build Feynman calculus.
Time to do a theory which corresponds to nature. Chapter 7 of Griffiths introduces QED, including the Feynman rules for the allowed vertices, etc. of QED.
This is the easiest and best understood of the QFTs in the Standard Model. It is the basis for everything that came later.
The first big difference from our toy model is that QED is a theory of spin-1/2 and spin-1 particles.

The Schrödinger Equation

If we start with the classical energy-momentum relationship:

$$\frac{\boldsymbol{p}^2}{2m} + V = E$$

and write the operators

$$\boldsymbol{p} \rightarrow \frac{\hbar}{i}\nabla, \; E \rightarrow i\hbar\frac{\partial}{\partial t}$$

We get the Schrödinger Equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

The Klein-Gordon Equation

The Schrödinger equation starts from a momentum-energy condition which is relativistically incorrect. What happens if we instead start from $E^2 - p^2c^2 = m^2c^4$ or from

$$p^\mu p_\mu - m^2c^2 = 0$$

Now write momentum as an operator

$$p_\mu \to i\hbar\partial_\mu$$

where

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}, \partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}, \partial_3 = \frac{\partial}{\partial z}$$

Substituting

The Klein-Gordon Equation

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi$$

The Dirac Equation

Unfortunately, the Klein-Gordon equation is second-order in time (leading to negative energy solutions and strange probability densities). Dirac wanted an equation which was first-order in time. So, how about we take

$$p^\mu p_\mu - m^2c^2 = 0$$

and factor it into two linear equations? OK, if we take the example of a particle at rest: $\boldsymbol{p} = 0$ then

$$\begin{aligned} (p^0)^2 - m^2c^2 &= 0 \\ (p^0 - mc)(p^0 + mc) &= 0 \\ &\Rightarrow (p^0 - mc) = 0, \textit{ or } (p^0 + mc) = 0 \end{aligned}$$

Either of these equations is linear in time and satisfies the relativistic energy-momentum relation.

The Dirac Equation

Extending this factorization to moving particles is not so easy. Let's write

$$(p^\mu p_\mu - m^2c^2) = (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc)$$

where β^κ and γ^λ are unknown coefficients. Expanding out the right side gives

$$(p^\mu p_\mu - m^2c^2) = \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - mc(\beta^\kappa - \gamma^\kappa)p_\kappa - m^2c^2$$

There are no linear p_κ terms on the left, so we need $\beta^\kappa = \gamma^\kappa$. Now, for left to equal right we need

$$p^\mu p_\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$$

The Dirac Equation

$$p^\mu p_\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$$

In other words:

$$\begin{aligned}(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 \quad = \quad & (\gamma^0)^2 (p^0)^2 \\ & +(\gamma^1)^2(p^1)^2 + (\gamma^2)^2(p^2)^2 + (\gamma^3)^2(p^3)^2 \\ & +(\gamma^0\gamma^1 + \gamma^1\gamma^0)p_0p_1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0)p_0p_2 \\ & +(\gamma^0\gamma^3 + \gamma^3\gamma^0)p_0p_3 + (\gamma^1\gamma^2 + \gamma^2\gamma^1)p_1p_2 \\ & +(\gamma^1\gamma^3 + \gamma^3\gamma^1)p_1p_3 + (\gamma^2\gamma^3 + \gamma^3\gamma^2)p_2p_3\end{aligned}$$

We have a problem. How can we make the LHS equal the RHS with all those nasty cross-terms? This cannot be solved if the γ's are complex numbers but it is possible if they are matrices!

The Dirac Equation

We need

$$(\gamma^0)^2 = 1,\ (\gamma^1)^2 = -1,\ (\gamma^2)^2 = -1,\ (\gamma^3)^2 = -1$$
$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0,\ \mu \neq \nu$$

This is usually written as

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

The γ matrixes are

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where σ^i are the Pauli matrices. So, our Dirac matrices look like 2x2 when they are 4x4.

Aside - Pauli Matrices

Remember the Pauli Matrices?

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Equation

What we were trying to do is we wanted to factorize:

$$(p^\mu p_\mu - m^2c^2) = (\gamma^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = 0$$

So, let's choose one of these two linear equations

$$\gamma^\mu p_\mu - mc = 0$$

and give it a name

The Dirac Equation

$$i\hbar\gamma^\mu \partial_\mu \psi - mc\psi = 0$$

or

$$(i\hbar\gamma^\mu \partial_\mu - mc)\psi = 0$$

Slash Notation

A common notation (though not in Griffiths at this stage) is Feynman slash notation

$$\gamma^{\mu} q_{\mu} = \not{q}$$

Dirac Equation in natural units and slash notation I get

$$\begin{aligned} \not{p} - m &= 0 \\ (i\not{\partial} - m)\psi &= 0 \end{aligned}$$

Spinors

We should note that ψ is now a 4-component Dirac spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

This is a column matrix, is not a 4-vector. We will discuss its Lorentz properties later.

Solutions to the Dirac Equation

Now that we have a Dirac equation, we need to find some solutions to it. Start by looking at solutions which are independent of position:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial z} = 0$$

This means that $\boldsymbol{p} = 0$ and the Dirac equation now has only the zeroth component:

$$\frac{i\hbar}{c}\gamma^0 \frac{\partial \psi}{\partial t} - mc\psi = 0$$

We will split the spinor into 2-component pieces:

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix},\ \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},\ \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

Solutions to the Dirac Equation

So, let's write our Dirac equation as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_A}{\partial t} \\ \frac{\partial \psi_B}{\partial t} \end{pmatrix} = -i \frac{mc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

In other words, we have 2 equations

$$\frac{\partial \psi_A}{\partial t} = -i \left(\frac{mc^2}{\hbar} \right) \psi_A, \; -\frac{\partial \psi_B}{\partial t} = -i \left(\frac{mc^2}{\hbar} \right) \psi_B$$

Solutions to the Dirac Equation

We have solutions of the form

$$\psi_A(t) = e^{-i(mc^2/\hbar)t}\psi_A(0),\ \psi_B(t) = e^{+i(mc^2/\hbar)t}\psi_B(0)$$

The Schrödinger equation has time-dependent solutions like:

$$e^{-iEt/\hbar}$$

This implies that we have ψ_A as a solution with energy mc^2 while ψ_B is a solution with energy $-mc^2$. More negative energy solutions!

Solutions to the Dirac Equation

- Dirac suggested that all possible negative states were already filled by a “Dirac sea” of particles. The Pauli exclusion principle would then leave only positive energy states available.
- The excitation of a sea electron would leave a hole which would behave like a positive energy particle with positive charge. Eventually Dirac predicted the existence of the positron.
- Experimentalists had been seeing evidence of the positron for quite a while but had dismissed it as “unphysical”. It was discovered very quickly after Dirac’s prediction.
- ψ_A describes electrons and ψ_B describes positrons. Each is a 2-component spinor.

Solutions to the Dirac Equation

We have solutions for electron spin-up, spin-down and positron spin-up and spin-down:

$$\psi^{(1)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad , \quad \psi^{(2)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\psi^{(3)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad , \quad \psi^{(4)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Plane-Wave Solutions to the Dirac Equation

So far we have only considered position-independent solutions to the Dirac equation. Now we should also consider plane-wave solutions:

$$\psi(x) = ae^{-(i/\hbar)x\cdot p}u(p)$$

If we substitute this into the Dirac equation we get

$$(\gamma^\mu p_\mu - mc)u = 0$$

(the "momentum space Dirac equation"). Now, let's write this out in matrix notation:

$$\begin{aligned}\gamma^\mu p_\mu = \gamma^0 p^0 - \gamma \cdot \boldsymbol{p} \quad &= \quad \frac{E}{c}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \boldsymbol{p}\cdot\begin{pmatrix}0 & \sigma\\ -\sigma & 0\end{pmatrix}\\ &= \quad \begin{pmatrix}E/c & -\boldsymbol{p}\cdot\boldsymbol{\sigma}\\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -E/c\end{pmatrix}\end{aligned}$$

Plane-Wave Solutions to the Dirac Equation

So, we can write:

$$
\begin{aligned}
(\gamma^\mu p_\mu - mc)u &= \begin{pmatrix} \left(\frac{E}{c} - mc\right) & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & \left(-\frac{E}{c} - mc\right) \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{E}{c} - mc\right) u_A - \boldsymbol{p}\cdot\boldsymbol{\sigma}\, u_B \\ \boldsymbol{p}\cdot\boldsymbol{\sigma}\, u_A - \left(\frac{E}{c} + mc\right) u_B \end{pmatrix}
\end{aligned}
$$

Remember that each row here should equal zero to satisfy the Dirac equation.

Plane-Wave Solutions to the Dirac Equation

We can satisfy the Dirac equation by having

$$u_A = \frac{c}{E - mc^2}(\boldsymbol{p} \cdot \boldsymbol{\sigma})u_B, \; u_B = \frac{c}{E + mc^2}(\boldsymbol{p} \cdot \boldsymbol{\sigma})u_A$$

Substituting one into the other:

$$u_A = \frac{c^2}{E^2 - m^2c^4}(\boldsymbol{p} \cdot \boldsymbol{\sigma})^2 u_A$$

We can show that $(\boldsymbol{p} \cdot \boldsymbol{\sigma})^2 = \boldsymbol{p}^2 \cdot \mathbf{1}$ so

$$u_A = \frac{\boldsymbol{p}^2 c^2}{E^2 - m^2c^4} u_A$$

or

$$E^2 - m^2c^4 = \boldsymbol{p}^2 c^2$$

Plane-Wave Solutions to the Dirac Equation

So, the Dirac equation enforces the relativistic energy-momentum relationship. This gives us two solutions:

$$E = \pm\sqrt{m^2c^4 + p^2c^2}$$

one for particles and one for antiparticles. The particles are

$$u^{(1)} = N\begin{pmatrix} 1 \\ 0 \\ \frac{c(p_z)}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix},\ u^{(2)} = N\begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \end{pmatrix}$$

with $E = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2}$, $N = \sqrt{(|E| + mc^2)/c}$

Plane-Wave Solutions to the Dirac Equation

The antiparticles are

$$v^{(1)} = N\begin{pmatrix} \frac{c(p_x - ip_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix},\ v^{(2)} = N\begin{pmatrix} \frac{c(p_z)}{E+mc^2} \\ \frac{c(p_x + ip_y)}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

with $E = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2}$.
While the particle solutions satisfy the Dirac equation of the form

$$(\gamma^\mu p_\mu - mc)u = 0$$

the antiparticle solutions obey

$$(\gamma^\mu p_\mu + mc)v = 0$$

Spins of the Plane-Wave Solutions

If you recall, in chapter 4 we wrote down 2-spinors to describe spin-up and spin down and then wrote the spin as

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. Now generalize this to

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\Sigma},\ \boldsymbol{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

If (and only if) we orient the z axis along the direction of motion will the plane-wave solutions be eigenstates of S_z. $u^{(1)}, v^{(1)}$ are spin-up and $u^{(2)}, v^{(2)}$ are spin-down.

Transformation Properties of Spinors

Now the notation will get maximally confusing. Spinors are not 4-vectors, so how do they transform? The rule for a system moving with speed v in the x direction is:

$$\psi \to \psi' = S\psi$$

where S is not spin but rather a 4x4 matrix

$$S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}$$

with

$$\begin{aligned} a_\pm &= \pm\sqrt{\frac{1}{2}(\gamma \pm 1)} \\ \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

Making a Scalar with Spinors

You might expect that an easy way to make a scalar out of a spinor is

$$\psi^\dagger\psi = (\psi_1^*\psi_2^*\psi_3^*\psi_4^*)\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

but that doesn't work (it does not transform like a scalar)

$$\begin{aligned}(\psi^\dagger\psi)' &\rightarrow (S\psi)^\dagger(S\psi)\\ &\rightarrow \psi^\dagger(S^\dagger S)\psi\end{aligned}$$

and you can show that $S^\dagger S \neq 1$

The Adjoint Spinor

Note that to make an invariant out of a 4-vector we had to introduce a metric $g^{\mu\nu}$ which was not the unit matrix...it has some minus signs in it. We now introduce the adjoint spinor

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = (+\psi_1^* + \psi_2^* - \psi_3^* - \psi_4^*)$$

Since the quantity

$$S^\dagger \gamma^0 S = \gamma^0$$

then

$$\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$

is a Lorentz scalar.
Since

$$(\bar{\psi}\psi)' = (\psi')^\dagger \gamma^0 \psi' = \psi^\dagger S^\dagger \gamma^0 S \psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$$

The Legend of γ^5

Define an additional γ matrix by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Some interesting properties: $(\gamma^5)^2 = 1$ and γ^5 anticommutes with any other γ

$$\{\gamma^\mu, \gamma^5\} = 0 \Rightarrow \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$$

A γ^5 Scalar and Parity

We have succeeded in defining one Lorentz scalar so far: $\bar{\psi}\psi$, can we also make one with γ^5? Well $S^\dagger\gamma^0\gamma^5 S = \gamma^0\gamma^5$ so $\bar{\psi}\gamma^5\psi$ is also a Lorentz scalar. Is there a difference between $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$?? What happens under parity? Parity applied to ψ gives:

$$P\psi = \gamma^0\psi$$

So, we see:

$$\begin{aligned}
\bar{\psi}\psi &\rightarrow (P\psi)^\dagger\gamma^0(P\psi) \\
&\rightarrow \psi^\dagger(\gamma^0)^\dagger\gamma^0\gamma^0\psi \\
&\rightarrow \psi^\dagger(\gamma^0)^\dagger\psi \\
&\rightarrow \bar{\psi}\psi
\end{aligned}
\qquad
\begin{aligned}
\bar{\psi}\gamma^5\psi &\rightarrow (P\psi)^\dagger\gamma^0\gamma^5(P\psi) \\
&\rightarrow \psi^\dagger(\gamma^0)^\dagger\gamma^0\gamma^5\gamma^0\psi \\
&\rightarrow -\psi^\dagger(\gamma^0)^\dagger\gamma^5\psi \\
&\rightarrow -\bar{\psi}\gamma^5\psi
\end{aligned}$$

$\bar{\psi}\psi$ is a scalar and $\bar{\psi}\gamma^5\psi$ is a pseudoscalar.

There are 16 possible ways to make a product of the form $\psi_i^* \psi_j$. These can be added together in various linear combinations and these can be grouped into bilinear covariants

$\bar{\psi}\psi$	scalar	1 component
$\bar{\psi}\gamma^5\psi$	pseudoscalar	1 component
$\bar{\psi}\gamma^\mu\psi$	vector	4 components
$\bar{\psi}\gamma^\mu\gamma^5\psi$	pseudovector	4 components
$\bar{\psi}\sigma^{\mu\nu}\psi$	antisymmetric tensor	6 components

where

$$\sigma^{\mu\nu} \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{2}[\gamma^\mu\gamma^\nu]$$

Why is this Useful?

- We have a simple basis set $\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$ for any 4x4 matrix. We can always simplify any other combination of γ matrices.
- The tensorial and parity characteristics of each bilinear is evident. So later, when we can see the QED interaction Lagrangian

$$-eA_\mu \bar{\psi}\gamma^\mu\psi$$

  we'll know that it leads to a parity-conservation in the EM interaction mediated by a vector (i.e. spin-1) boson.
- To describe the parity-violating weak interaction we mix vector ($\bar{\psi}\gamma^\mu\psi$) and axial ($\bar{\psi}\gamma^\mu\gamma^5\psi$) interactions.

The Photon

- Maxwell's equation

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 4\pi J^\nu$$

where $\Box = \partial_\mu \partial^\mu$, $A^\nu = (\phi, \mathbf{A})$ and $J^\nu = (\rho, \mathbf{J})$

- $(\phi, \mathbf{A})$ are not uniquely determined and so we are allowed to make a gauge transformation $A^\mu \to A^\mu + \partial^\mu \lambda$
- We can demand the Lorentz condition $\partial_\mu A^\mu = 0$
- The Lorentz condition simplifies the Maxwell equations to

$$\Box A^\mu = 4\pi J^\mu$$

Another constraint

- Even with the Lorentz condition, we can make further gauge transformations of the form $A^\mu \to A^\mu + \partial^\mu \lambda$ without disturbing $\Box A^\mu = 4\pi J^\mu$ so long as $\Box \lambda = 0$.
- As a result, we can impose an additional constraint. We typically choose to set $A^0 = 0$ and thereby work in the *Coulomb gauge*:

$$\nabla \cdot \mathbf{A} = 0$$

Free Photons

- For a photon in free space ($J^\mu = 0$), the potential is given by $\Box A^\mu = 0$.
- The plane-wave solution is

$$A^\mu(x) = a e^{-ip\cdot x} \epsilon^\mu(p)$$

where ϵ^μ is the *polarization vector* and $p_\mu p^\mu = 0$.

- Although ϵ^μ has 4 components, not all are independent. The Lorentz condition requires that $p^\mu \epsilon_\mu = 0$ Furthermore, the Coulomb gauge implies that $\epsilon^0 = 0$ and $\epsilon \cdot \mathbf{p} = 0$
- Since ϵ is perpendicular to $\mathbf{p}$, the photon is *transversely polarized* and there are only 2 independent polarization states.

The Feynman Rules for QED

Finally, what we've all been waiting for: the Feynman rules for QED! The rules are going to be very similar to, but not identical to,the toy model. We will spend quite a bit of time on examples/applications of these rules in the next lectures.

Summary of Results - Electrons and Positrons

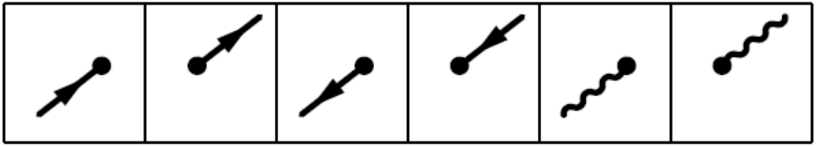
	Electron	Positron
Wave Fn.	$\psi(x) = ae^{-(i/\hbar)p\cdot x}u^{(s)}(p)$	$\psi(x) = ae^{(i/\hbar)p\cdot x}v^{(s)}(p)$
Dirac Eq.	$(\gamma^\mu p^\mu - mc)u = 0$	$(\gamma^\mu p^\mu + mc)v = 0$
Adjoints	$\bar{u} = u^\dagger\gamma^0$ $\bar{u}(\gamma^\mu p_\mu - mc) = 0$	$\bar{v} = v^\dagger\gamma^0$ $\bar{v}(\gamma^\mu p_\mu + mc) = 0$
Orthogonal	$\bar{u}^{(1)}u^{(2)} = 0$	$\bar{v}^{(1)}v^{(2)} = 0$
Normalized	$\bar{u}u = 2mc$	$\bar{v}v = -2mc$
Completeness	$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} =$ $(\gamma^\mu p_\mu + mc)$	$\sum_{s=1,2} v^{(s)}\bar{v}^{(s)} =$ $(\gamma^\mu p_\mu - mc)$

Summary of Results - Photons

	Photon
Wave Fn.	$A^{\mu}(x) = ae^{-(i/\hbar)p\cdot x}\epsilon^{\mu}_{(s)}$
Lorentz Condition	$\epsilon^{\mu}p_{\mu} = 0$
Othogonal	$\epsilon^{\mu}_{(1)}\epsilon_{\mu(2)} = 0$
Normalized	$\epsilon^{\mu *}\epsilon_{\mu} = 1$
Coulomb Gauge	$\epsilon^{0} = 0,\ \boldsymbol{\epsilon}\cdot\boldsymbol{p} = 0$
Completeness	$\sum_{s=1,2}(\epsilon_{(s)})_i(\epsilon^{*}_{(s)})_j = \delta_{ij} - \hat{p}_i\hat{p}_j$

The Feynman Rules for QED

1. Draw the Feynman diagrams with the minimum number of vertices
2. Label the momenta (p_1, p_2, etc.) but **also add spin labels** and add q's to the internal lines as in the toy model.
3. External lines contribute:



e^- in	e^- out	e^+ in	e^+ out	γ in	γ out
u	$\bar{u}$	$\bar{v}$	v	ϵ_μ	ϵ_μ^*

Feynman Rules

4. Each vertex contributes a factor:

$$ig_e\gamma^\mu$$

where g_e is related to the charge of the positron or the fine structure constant $g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha}$

5. Each internal line contributes a factor:

$$\textit{electrons and positrons} \quad : \quad \frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$$

$$\textit{photons} \quad : \quad \frac{-ig_{\mu\nu}}{q^2}$$

6. Conservation of energy and momentum makes each vertex contribute

$$(2\pi)^4\delta^4(k_1 + k_2 + k_3)$$

particles coming in are positive and going out are negative (antiparticles are opposite).

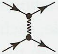
Feynman Rules

7. Integrate over internal momenta. For each internal momentum q write a factor
$$\frac{d^4q}{(2\pi)^4}$$
and integrate.
8. Cancel the delta function. Erase the term that looks like
$$(2\pi)^4\delta^4(p_1 + p_2 + \cdots - p_n)$$
Multiply by i, what remains is $\mathcal{M}$
9. Antisymmetrization: Include a minus sign between diagrams which differ only in the exchange of two incoming (or outgoing) electrons (or positrons) or of an incoming electron with an outgoing positron (or vice versa).

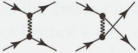
QED Catalogue

Second-order processes

Elastic

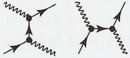


Electron–muon scattering($e + \mu \to e + \mu$)
(Mott scattering($M \gg m$) $\Rightarrow$ Rutherford scattering($v \ll c$))



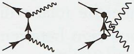
Electron–electron scattering($e^- + e^- \to e^- + e^-$)
(Møller scattering)

Electron–positron scattering($e^- + e^+ \to e^- + e^+$)
(Bhabha scattering)

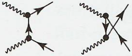


Compton scattering($\gamma + e^- \to \gamma + e^-$)

Inelastic



Pair annihilation($e^- + e^+ \to \gamma + \gamma$)

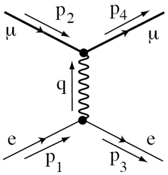


Pair production($\gamma + \gamma \to e^- + e^+$)

Most important third-order process

$\Rightarrow$ Anomalous magnetic moment of electron

Electron-Muon Scattering



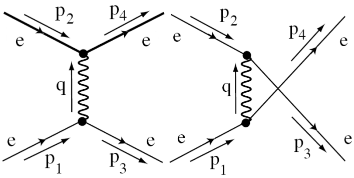
We need to proceed "backwards" along the fermion lines and build-up an integral.
Starting from p_3 line, vertex, p_1 line, photon propogator, p_4 line, vertex, p_2 line and δ functions at each vertex.

$$(2\pi)^4 \int [\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)u^{(s_1)}(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}^{(s_4)}(p_4)(ig_e\gamma^\nu)u^{(s_2)}(p_2)] \\ \times \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4)d^4q$$

Integrating (with deltas) gives:

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)]$$

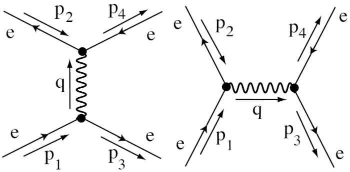
Electron-Electron Scattering



The first diagram is just like the previous page. The second is the first with an exchange of p_3 and p_4, so we get a negative sign between them:

$$\begin{aligned}\mathcal{M} &= -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)] \\ &+ \frac{g_e^2}{(p_1 - p_4)^2}[\bar{u}(4)\gamma^\mu u(1)][\bar{u}(3)\gamma_\mu u(2)]\end{aligned}$$

Electron-Positron Scattering



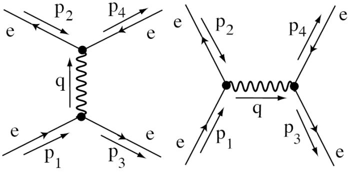
Now dealing with antiparticles. Now we must remember *v*'s instead of *u*'s and that backwards is forwards. The first diagram gives

$$(2\pi)^4 \int [\bar{u}(3)(ig_e\gamma^\mu)u(1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{v}(2)(ig_e\gamma^\nu)v(4)] \\ \times \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4)d^4q$$

Which gives the amplitude

$$\mathcal{M}_1 = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{v}(2)\gamma_\mu v(4)]$$

Electron-Positron Scattering



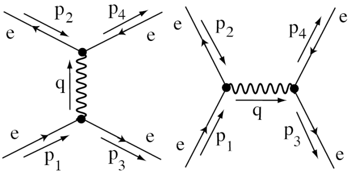
The second diagram gives

$$(2\pi)^4 \int [\bar{u}(3)(ig_e\gamma^\mu)v(4)] \frac{-ig_{\mu\nu}}{q^2} [\bar{v}(2)(ig_e\gamma^\nu)u(1)]$$
$$\times \delta^4(q - p_3 - p_4)\delta^4(p_1 + p_2 - q)d^4q$$

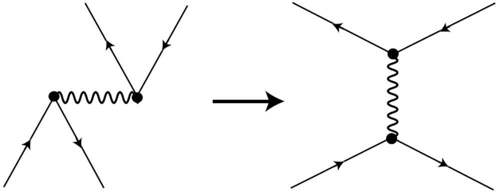
Which gives the amplitude

$$\mathcal{M}_2 = -\frac{g_e^2}{(p_1 + p_2)^2} [\bar{u}(3)\gamma^\mu v(4)][\bar{v}(2)\gamma_\mu u(1)]$$

Electron-Positron Scattering

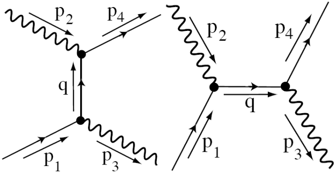


Do we add or subtract these amplitudes? Interchaging the incoming positron and outgoing electron in the second diagram:



Subtract them. $\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$

Compton Scattering



OK, so this one has an electron internal line an 2 photon external lines. The first diagram yields

$$(2\pi)^4 \int \epsilon_\mu(2) \left[\bar{u}(4)(ig_e\gamma^\mu) \frac{i(\not{q} + mc)}{(q^2 - m^2c^2)} (ig_e\gamma^\nu) u(1) \right]$$
$$\times \epsilon_\nu(3)^* \delta^4(p_1 - p_3 - q) \delta^4(p_2 + q - p_4) d^4q$$

Compton Scattering

Giving amplitudes

$$\mathcal{M}_1 = \frac{g_e^2}{(p_1 - p_3)^2 - m^2c^2}[\bar{u}(4)\not{\epsilon}(2)(\not{p}_1 - \not{p}_3 + mc)\not{\epsilon}(3)^* u(1)]$$
$$\mathcal{M}_2 = \frac{g_e^2}{(p_1 + p_2)^2 - m^2c^2}[\bar{u}(4)\not{\epsilon}(3)^*(\not{p}_1 + \not{p}_2 + mc)\not{\epsilon}(2) u(1)]$$

Then add 'em as the two diagrams do not differ by the interchange of fermions.

So, what do we do with $\mathcal{M}$??

The Feynman Rules have served us well. Give me a diagram and I'll show you an $\mathcal{M}$. This is not the end of the story though. It is all well-and-good to say that we can derive some ugly expression for $\mathcal{M}$ and can then square it and plug into a cross section formula, but how do you actually **do** it? Remember that our expressions for $\mathcal{M}$ are filled with spinors. I guess we need to specify the spin-states of the incoming and outgoing particles and get one $\mathcal{M}$ for each combination!!!

Sometimes this is exactly what you want to do. For example, you may wish to do an experiment with polarized beams and/or targets. However, most of the time we want to average over initial spin configurations and sum over final spin configurations.

$$\langle|\mathcal{M}|^2\rangle \equiv \textit{average over initial, sum over final spins}$$

Casimir's Trick and Trace Theorems

To calculate $\langle|\mathcal{M}|^2\rangle$ you could first calculate the amplitudes of each configuration and then combine....or use some tricks to do it all at once. Consider the electron-muon scattering process we saw earlier:

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1 - p_3)^4}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)][\bar{u}(3)\gamma^\nu u(1)]^*[\bar{u}(4)\gamma_\nu u(2)]^*$$

This (and other) element shows that we will be doing lots of algebra with quantities of the form

$$G \equiv [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^*$$

So, let's develop some simplifying tricks for these.

Casimir's Trick and Trace Theorems

Let's look at the conjugate (use definition of $\bar{u}$ and change order when conjugating):

$$[\bar{u}(a)\Gamma_2 u(b)]^* = [u(a)^\dagger \gamma^0 \Gamma_2 u(b)]^\dagger = u(b)^\dagger \Gamma_2^\dagger \gamma^{0\dagger} u(a)$$

Now, we can insert pairs of γ^0's since $(\gamma^0)^2 = 1$ and we know that $\gamma^{0\dagger} = \gamma^0$ so

$$\begin{aligned} [\bar{u}(a)\Gamma_2 u(b)]^* &= [u(b)^\dagger]\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 u(a) \\ &= \bar{u}(b)\bar{\Gamma}_2 u(a) \end{aligned}$$

where $\bar{\Gamma}_2 = \gamma^0\Gamma_2^\dagger\gamma^0$. Thus

$$G = [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(b)\bar{\Gamma}_2 u(a)]$$

Casimir's Trick and Trace Theorems

The completeness relation says:

$$\sum_{s_i=1,2} u_i^{s_i}\bar{u}_i^{s_i} = (\not{p}_i + m_i c)$$

So, we can sum over the spins of particle "b" in G

$$G = [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(b)\bar{\Gamma}_2 u(a)]$$

and get

$$\begin{aligned}
\sum_{b\ spins} G &= \bar{u}(a)\Gamma_1\left\{\sum_{s_b=1,2} u^{(s_b)}(p_b)\bar{u}^{(s_b)}(p_b)\right\}\bar{\Gamma}_2 u(a) \\
&= \bar{u}(a)\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2 u(a) \\
&= \bar{u}(a)Qu(a)
\end{aligned}$$

where $Q \equiv \Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2$

Casimir's Trick and Trace Theorems

Now do the same summation for particle "a"

$$\sum_{a \text{ spins}} \sum_{b \text{ spins}} G = \sum_{s_a=1,2} \bar{u}^{(s_a)}(p_a) Q u^{(s_a)}(p_a)$$

Writing out the matrix multiplication explicitly

$$\begin{aligned}
\sum_{s_a=1,2} \sum_{i,j=1}^{4} \bar{u}^{(s_a)}(p_a)_i Q_{ij} u^{(s_a)}(p_a)_j &= \sum_{i,j=1}^{4} Q_{ij} \left\{ \sum_{s_a=1,2} u^{(s_a)}(p_a) \bar{u}^{(s_a)}(p_a) \right\}_{ji} \\
&= \sum_{i,j=1}^{4} Q_{ij} (\not{p}_a + m_a c)_{ji} \\
&= \sum_{i=1}^{4} [Q(\not{p}_a + m_a c)]_{ii} \\
&= Tr(Q(\not{p}_a + m_a c))
\end{aligned}$$

where 'Tr' denotes the **trace** of the matrix which is the sum of the diagonal elements.

Casimir's Trick

Casimir's Trick

$$\sum_{\textit{all spins}} [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^* = Tr[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]$$

Notice that there are “NO SPINORS LEFT!!”- once we do summation over spins, all that remains is to multiply matrices and take the trace.

If either u is replaced by a v, the corresponding mass on the right-hand side switches sign.

Trace Theorems

So, using Casimir's trick we reduce everything to the manipulation of γs and the calculation of traces. Let's collect some useful trace theorems and other handy relations.

1. $Tr(A+B) = Tr(A) + Tr(B)$
2. $Tr(\alpha A) = \alpha Tr(A)$
3. $Tr(AB) = Tr(BA)$
4. $g_{\mu\nu} g^{\mu\nu} = 4$
5. $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$

5′ $\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b$

6. $\gamma_\mu \gamma^\mu = 4$

Trace Theorems

7. $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$

7′ $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$

8. $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$

8′ $\gamma_\mu \not{a} \not{b} \gamma^\mu = 4a \cdot b$

9. $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\lambda \gamma^\nu$

9′ $\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2\not{c} \not{b} \not{a}$

10. The trace of a product of an odd number of γs is zero

11. $Tr(1) = 4$

12. $Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

12′ $Tr(\not{a} \not{b}) = 4a \cdot b$

13. $Tr(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda})$

13′ $Tr(\not{a} \not{b} \not{c} \not{d}) = 4(a \cdot b\, c \cdot d - a \cdot c\, b \cdot d + a \cdot d\, b \cdot c)$

Trace Theorems

Note that γ^5 is the product of an even number of γ^0 so multiplying by an odd number of γs the trace is zero.

14. $Tr(\gamma^5) = 0$
15. $Tr(\gamma^5 \gamma^\mu \gamma^\nu) = 0$

15′ $Tr(\gamma^5 \not{a} \not{b}) = 0$

16. $Tr(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$

16′ $Tr(\gamma^5 \not{a} \not{b} \not{c} \not{d}) = 4i\epsilon^{\mu\nu\lambda\sigma} a_\mu b_\nu c_\lambda d_\sigma$

with

$$\epsilon^{\mu\nu\lambda\sigma} = \left\{ \begin{array}{l} -1 \textit{ if } \mu\nu\lambda\sigma \textit{ is an even permutation of } 0123 \\ +1 \textit{ if } \mu\nu\lambda\sigma \textit{ is an odd permutation of } 0123 \\ 0 \textit{ if any 2 indices are identical} \end{array} \right\}$$

Example: Electron-Muon Scattering

Let's try a simple trace example: electron-muon scattering:

$$Tr[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)]$$

This gives 4 terms

$$Tr[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 + \gamma^\mu mc \gamma^\nu mc + \gamma^\mu mc \gamma^\nu \not{p}_3 + \gamma^\mu \not{p}_1 \gamma^\nu mc]$$

The last two terms are the product of an odd number of gammas and so are zero. Then

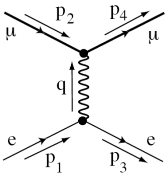
$$\begin{aligned} &= Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) + m^2c^2 Tr(\gamma^\mu \gamma^\nu) \\ &= 4(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - (p_1 \cdot p_3) g^{\mu\nu}) + 4m^2c^2 g^{\mu\nu} \end{aligned}$$

Another Example: Contract 2 Traces

Let's try (try it on your own):

$$\begin{aligned}
A &= Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2) Tr(\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2) \\
&= 4[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1 \cdot p_2) g^{\mu\nu}] \\
&\quad \times 4[p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - (p_1 \cdot p_2) g_{\mu\nu}] \\
&= 16[2p_1^2 p_2^2 + 2(p_1 \cdot p_2)^2 + 4(p_1 \cdot p_2)^2 - 4(p_1 \cdot p_2)^2] \\
&= 32[m_1^2 m_2^2 + (p_1 \cdot p_2)^2]
\end{aligned}$$

Electron-Muon Scattering



Our first example application of the Feynman Rules for QED was electron-muon scattering:

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^{\mu}u(1)][\bar{u}_4\gamma_{\mu}u(2)]$$

Let's evaluate the traces.

Electron-Muon Scattering

$$
\begin{aligned}
\mathcal{M} &= -\frac{g_e^2}{(p_1-p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}_4\gamma_\mu u(2)] \\
\langle|\mathcal{M}|^2\rangle &= \frac{g_e^4}{4(p_1-p_3)^4} Tr[\gamma^\mu(\not{p}_1+mc)\gamma^\nu(\not{p}_3+mc)] \\
&\times\; Tr[\gamma_\mu(\not{p}_2+Mc)\gamma_\nu(\not{p}_4+Mc)] \\
&= \frac{g_e^4}{4(p_1-p_3)^4}[4(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (m^2c^2 - p_1\cdot p_3)g^{\mu\nu})] \\
&\times\; [4(p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + (M^2c^2 - p_2\cdot p_4)g_{\mu\nu})] \\
&= \frac{4g_e^4}{(p_1-p_3)^4}\{2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) \\
&+\; 2m^2c^2(p_2\cdot p_4) + 2M^2c^2(p_1\cdot p_3) - 4(p_1\cdot p_3)(p_2\cdot p_4) \\
&+\; 4(m^2c^2 - p_1\cdot p_3)(M^2c^2 - p_2\cdot p_4)\}
\end{aligned}
$$

Electron-Muon Scattering

$$\langle|\mathcal{M}|^2\rangle = \frac{8g_e^4}{(p_1-p_3)^4}\{(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3) \\ -m^2c^2(p_2\cdot p_4)-M^2c^2(p_1\cdot p_3)+2m^2M^2c^4$$

Remember the Mandelstam Variables? Let's write them down neglecting mass terms

$$\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2 \simeq 2p_1\cdot p_2 = 2p_3\cdot p_4 \\ t &= (p_1-p_3)^2 = (p_2-p_4)^2 \simeq -2p_1\cdot p_3 = -2p_2\cdot p_4 \\ u &= (p_1-p_4)^2 = (p_2-p_3)^2 \simeq -2p_2\cdot p_3 = -2p_1\cdot p_4 \end{aligned}$$

So, we could write our amplitude (ignoring mass terms) using these:

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{8g_e^4}{(p_1-p_3)^4}[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)] \\ &= \frac{8g_e^4}{t^2}\frac{1}{4}(s^2+u^2) = 2g_e^4(\frac{s^2+u^2}{t^2}) \end{aligned}$$

Mott Scattering

Another name for electron-muon scattering is “Mott scattering”. Assume that

$$M \gg m, E, \boldsymbol{p}$$

and that the muon recoil can be neglected. The differential cross section is:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi Mc}\right)^2 \langle |\mathcal{M}|^2 \rangle$$

Mott Scattering

The 4-momenta are then

$$p_1 = (E/c, \boldsymbol{p}_1),\; p_2 = (Mc, \boldsymbol{0}),\; p_3 \simeq (E/c, \boldsymbol{p}_3),\; p_4 \simeq (Mc, \boldsymbol{0})$$

And the momentum transfer is (calling $|\boldsymbol{p}| = |\boldsymbol{p}_1| = |\boldsymbol{p}_3|$)

$$\begin{aligned}
(p_1 - p_3)^2 &= (0, \boldsymbol{p}_1 - \boldsymbol{p}_3)^2 \\
&= -\boldsymbol{p}_1^2 - \boldsymbol{p}_3^2 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_3 \\
&= -2\boldsymbol{p}^2(1 - \cos\theta) \\
&= -4\boldsymbol{p}^2 \sin^2\frac{\theta}{2}
\end{aligned}$$

Let's also evaluate each product in the amplitude:

$$\begin{aligned}
(p_1 \cdot p_2)(p_3 \cdot p_4) &= (p_1 \cdot p_4)(p_2 \cdot p_3) = (ME)^2 \\
(p_2 \cdot p_4) &= (Mc)^2
\end{aligned}$$

Mott Scattering

OK, let's plug those in

$$
\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{8g_e^4}{(p_1-p_3)^4}[(p_1\cdot p_2)(p_3\cdot p_4)+(p1\cdot p_4)(p_2\cdot p_3)] \\
&\quad -m^2c^2(p_2\cdot p_4)-M^2c^2(p_1\cdot p_3)+2m^2M^2c^4 \\
&= \frac{g_e^4}{2\boldsymbol{p}^4\sin^4\theta/2}[2M^2E^2-m^2M^2c^4+2(mMc^2)^2 \\
&\quad -(Mc)^2(m^2c^2+2\boldsymbol{p}^2\sin^2\theta/2)] \\
&= \left(\frac{g_e^2Mc}{\boldsymbol{p}^2\sin^2\theta/2}\right)^2\left(\frac{E^2}{c^2}-\boldsymbol{p}^2\sin^2\theta/2\right) \\
&= \left(\frac{g_e^2Mc}{\boldsymbol{p}^2\sin^2\theta/2}\right)^2\left(m^2c^2+\boldsymbol{p}^2\cos^2\theta/2\right) \\
\frac{d\sigma}{d\Omega} &= \left(\frac{\alpha\hbar}{2\boldsymbol{p}^2\sin^2\theta/2}\right)^2\left[(mc)^2+\boldsymbol{p}^2\cos^2\theta/2\right]
\end{aligned}
$$

Rutherford Scattering Limit

In the non-relativistic limit we get Rutherford scattering

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{2\boldsymbol{p}^2 \sin^2\theta/2}\right)^2 \left[(mc)^2 + \boldsymbol{p}^2 \cos^2\theta/2\right]$$

Now substitute

$$\begin{aligned} m^2c^2 + \boldsymbol{p}^2\cos^2\theta/2 &\rightarrow m^2c^2 \\ \boldsymbol{p}^2 &\rightarrow 2mE\,(E = kinetic) \\ \alpha &\rightarrow q_1 q_2 \end{aligned}$$

and we get

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{2mv^2 \sin^2\theta/2}\right)^2$$

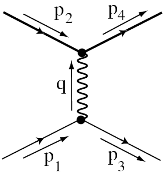
Renormalization

We have already discussed renormalization in a qualitative way (remember dielectric, vaccum polarization, bare mass, measured mass, running couplings).

We cannot discuss it in detailed quantitative way in this course, but we can do **something** to get a better quantitative feel now that we know the QED Feynman rules.

Renormalization

We have considered electron-muon scattering at lowest order



with scattering amplitude

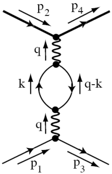
$$\mathcal{M} = -g_e^2[\bar{u}(3)\gamma^\mu u(1)]\frac{g_{\mu\nu}}{q^2}[\bar{u}(4)\gamma^\nu u(2)]$$

with $q = p_1 - p_3$.

Renormalization

$$\mathcal{M} = -g_e^2[\bar{u}(3)\gamma^\mu u(1)]\frac{g_{\mu\nu}}{q^2}[\bar{u}(4)\gamma^\nu u(2)]$$

Of course, this is only leading order. What about our famous "vacuum polarization" diagram?



$$\begin{aligned}\mathcal{M} &= -g_e^2[\bar{u}(3)\gamma^\mu u(1)]\left\{\int \frac{d^4k}{(2\pi)^4}\frac{Tr[\gamma_\mu(\not{k}+mc)\gamma_\nu(\not{q}-\not{k}+mc)]}{(k^2-m^2c^2)((q-k)^2-m^2c^2)}\right\} \\ &\quad \times[\bar{u}(4)\gamma^\nu u(2)]\end{aligned}$$

Effectively we have modified the propagator:

$$\frac{g_{\mu\nu}}{q^2} \rightarrow \frac{g_{\mu\nu}}{q^2} - \frac{i}{q^4} I_{\mu\nu}$$

We are hiding a bunch of complexity with

$$I_{\mu\nu} = -g_e^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr[\gamma_\mu(\not{k} + mc)\gamma_\nu(\not{q} - \not{k} + mc)]}{(k^2 - m^2c^2)((q-k)^2 - m^2c^2)}$$

which is divergent as $k \rightarrow \infty$.
Diagrams with closed loops are just a pain.

We can reduce this integral to

$$I(q^2) = \frac{g_e^2}{12\pi^2}\left\{\int_{m^2}^{\infty}\frac{dx}{x} - 6\int_0^1 z(1-z)\ln\left(1 - \frac{q^2}{m^2c^2}z(1-z)\right)dz\right\}$$

by pure magic. The benefit of this magic is to isolate the divergence in the first term. To “handle” the divergence we introduce a mass cutoff M:

$$\int_{m^2}^{\infty}\frac{dx}{x} \rightarrow \int_{m^2}^{M^2}\frac{dx}{x} = \ln\frac{M^2}{m^2}$$

The second (finite) integral is given the symbol f. So, we now have

$$I(q^2) = \frac{g_e^2}{12\pi^2}\left\{\ln\left(\frac{M^2}{m^2}\right) - f\left(\frac{-q^2}{m^2c^2}\right)\right\}$$

Renormalization

OK, we finally substitute this expression for *I* back in and get the amplitude

$$\mathcal{M} = -g_e^2[\bar{u}(3)\gamma^\mu u(1)]\frac{g_{\mu\nu}}{q^2}\left\{1 - \frac{g_e^2}{12\pi^2}\left[\ln\left(\frac{M^2}{m^2}\right) - f\left(\frac{-q^2}{m^2c^2}\right)\right]\right\} \times[\bar{u}(4)\gamma^\nu u(2)]$$

Finally we renormalize. Let's introduce a new, effective, coupling constant

$$g_R \equiv g_e\sqrt{1 - \frac{g_e^2}{12\pi^2}\ln\left(\frac{M^2}{m^2}\right)}$$

and we can write the amplitude in terms of it

$$\mathcal{M} = -g_R^2[\bar{u}(3)\gamma^\nu u(1)]\frac{g_{\mu\nu}}{q^2}\left\{1 + \frac{g_R^2}{12\pi^2}f\left(\frac{-q^2}{m^2c^2}\right)\right\}[\bar{u}(4)\gamma^\nu u(2)]$$

Renormalization

$$\mathcal{M} = -g_R^2[\bar{u}(3)\gamma^\nu u(1)]\frac{g_{\mu\nu}}{q^2}\left\{1 + \frac{g_R^2}{12\pi^2}f\left(\frac{-q^2}{m^2c^2}\right)\right\}[\bar{u}(4)\gamma^\nu u(2)]$$

Things to note:

- There is no M in the equation, so infinities are gone. The cutoff scale does not even appear! The price we pay is to replace g_e by g_R, but it turns out that g_R is what we measure anyway.
- There is still a finite correction term from the higher order diagrams which depends on q^2. This can also be put into the coupling constant if you want but that makes the coupling "run"

$$g_R(q^2) = g_R(0)\sqrt{1 + \frac{g_R(0)^2}{12\pi^2}f\left(\frac{-q^2}{m^2c^2}\right)}$$

This means that the effective charge of an electron varies depending on the momentum transferred in the collision - screening! At LEP α was not $1/137$, rather it was $1/128$!